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Fractional coloring conjecture for shadow tomography

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Shadow tomography

Goal: given ϱ and a set of observables $P \in S$, estimate $\mathrm{tr}(\varrho P)$.

More precisely, when talking about *shadow tomography*:

Definition (Shadow tomography)

The shadow tomography task for a set S of observables is as follows. We are given copies of an unknown state ϱ, and our goal is to output estimates y_P such that (with high probability) we have $|y_P - \mathrm{tr}(P\varrho)| \leq \epsilon$ for all $P \in S$.

King, et al. Triply efficient shadow tomography. PRX Quantum, 6(1):010336, 2025

$\longrightarrow$ focus here on Pauli observables

Naive strategy: coloring

Each commuting set can be measured simultaneously.

Formalize: Let G be the anti-commutativity graph of the Pauli set, where two vertices are connected, if and only if the corresponding operators anti-commute

$$\begin{aligned} i \sim j \qquad & \text{if} \quad P_i P_j = -P_j P_i \\ i \nsim j \qquad & \text{if} \quad P_i P_j = P_j P_i \end{aligned}$$

- Color graph with k colors, so that any two adjacent vertices have a different color.
- k sets of mutually commuting operators.
- The smallest number of colors needed for such coloring is the chromatic number χ. (NP-hard)

$\longrightarrow$ The number of samples for naive tomography scales with χ.

Better strategy: fractional coloring

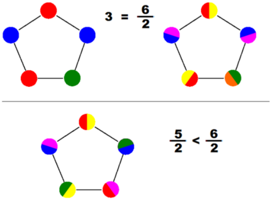
- probabilistically sample independent sets (sets of mutually commuting elements) with some overlaps.
- The natural relaxation is the *fractional chromatic number*

The *fractional chromatic number* is a linear-programming relaxation of the chromatic number,

$$\chi_{\mathrm{f}}(G) = \min \left\{ \sum_{I \in \mathcal{I}(G)} w_I : w_I \geq 0, \quad \sum_{I \,:\, v \in I} w_I \geq 1 \text{ for every } v \in V(G) \right\}.$$

where $\mathcal{I}(G)$ is the set of indepedent sets of G.

Each independent set receives a nonnegative weight and every vertex must receive total weight at least one.



Fractional coloring strategy

- With the fractional coloring strategy, the number of samples scales with the fractional coloring number χ_f.
- Recall: We actually only care up to precision ϵ:

Definition (Shadow tomography)

The shadow tomography task for a set S of observables is as follows. We are given copies of an unknown state ϱ, and our goal is to output estimates y_P such that (with high probability) we have $|y_P - \mathrm{tr}(P\varrho)| \leq \epsilon$ for all $P \in S$.

King et al showed that the number of samples scales as:

$$\begin{aligned} \text{samples:} \quad & \Omega\big(\chi_f \log(|S|)/\epsilon^2\big)\,, \\ \text{runtime:} \quad & \Omega\big((T + n^3)\chi_f(\log|S|)/\epsilon^2\big)\,. \end{aligned}$$

Fractional coloring conjecture

In what follows:

$$B_\epsilon(\varrho_m) = \{P \in \mathcal{P}_n : |\operatorname{tr}(\varrho_m P)| \geq \epsilon\},$$

(The set of observables we care about, with sufficient large expecation values.)

Conjecture (Conjecture 13, King et al.)

Let ϱ be an n-qubit state, $\epsilon \in (0, 1)$, and let $B_\epsilon \subseteq \mathcal{P}_n$ be the set of all Paulis P such that $|\operatorname{tr}(\varrho P)| \geq \epsilon$. There is a fractional coloring of the commutation graph $G(B_\epsilon)$ of size $O(1/\epsilon^2)$.

King, et al. Triply efficient shadow tomography. PRX Quantum, 6(1):010336, 2025

- If true, there would exist a "triply efficient" shadow tomography protocol for the set of all Paulis.
- We show that this conjecture is false.

Fractional coloring conjecture

Rephrased as an inequality, the Conjecture asserts:
there exists a constant C, such that for every state ϱ and error ϵ

$$\chi_f(B_\epsilon(\varrho)) \cdot \epsilon^2 \leq C\,,$$

where χ_f is the fractional chromatic number of the anticommutation graph of $B_\epsilon(\varrho)$.

- Thus we need to show that no such finite constant C exists.

Amplification

Conjecture: There exists a constant C, such that for all states and observables

$$\chi_f(B_\epsilon(\varrho)) \cdot \epsilon^2 \leq C\,,$$

- Construct a family of states and observables for which C must necessarily be unbounded.
- We use the idea of amplification throught the lexicographic product. This way the lhs $\longrightarrow \infty$.

Lexicographic product

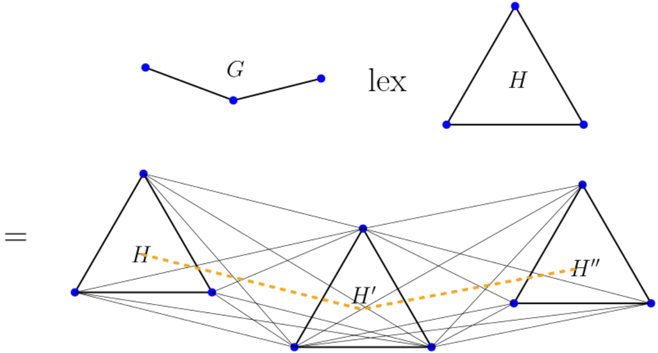
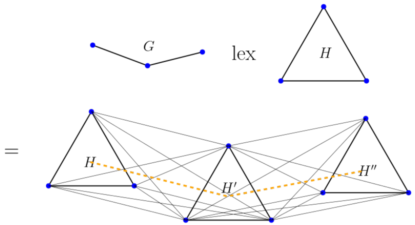


Figure: The lexicographic product of the line graph with the triangle, $L_3 \operatorname{lex} C_3$. At every vertex of G (a branch), place a copy of H (a leaf). Then connect the vertices among different leaves if and only if their branches are connected.

Lex product of observables

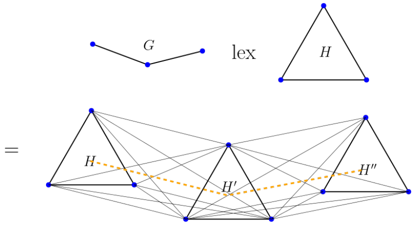


Let $G = L_3$ and $H = C_3$. The graph G is realized by $T = \{XI, ZZ, IX\}$, and H is realized by $R = \{X, Y, Z\}$. A set realizing $(G \operatorname{lex} H)$ is

leaf H:	$\mathrm{XI} \,\vert\, \mathrm{XII}$,	$\mathrm{XI} \,\vert\, \mathrm{YII}$,	$\mathrm{XI} \,\vert\, \mathrm{ZII}$,
leaf H':	$\mathrm{ZZ} \,\vert\, \mathrm{IXI}$,	$\mathrm{ZZ} \,\vert\, \mathrm{IYI}$,	$\mathrm{ZZ} \,\vert\, \mathrm{IZI}$,
leaf H'':	$\mathrm{IX} \,\vert\, \mathrm{IIX}$,	$\mathrm{IX} \,\vert\, \mathrm{IIY}$,	$\mathrm{IX} \,\vert\, \mathrm{IIZ}$.

$\longrightarrow$ for every copy (leaf) use a separate tensor factor/register

Lex product of observables II



$$T \,\mathrm{lex}\, R \;=$$

$$\left\{ \overbrace{T_i}^{\text{Reg. 1}} \otimes \overbrace{I \otimes \cdots \otimes I \otimes \underbrace{R_j}_{i\text{'th tensor factor}} \otimes I \otimes \cdots \otimes I}^{\text{Reg. 2}} \;\middle|\; i = 1, \ldots, t\,;\; j = 1, \ldots, r \right\}.$$

Here T_i is placed on the first register, and R_j placed onto the i-th tensor factor of the second register, where the second register contains $|V(G)|$ internal tensor factors.

Lex product properties

It is known that

$$\chi_f(G \operatorname{lex} H) = \chi_f(G)\chi_f(G)\,, \qquad \text{(fract. coloring number)}$$
$$\alpha(G \operatorname{lex} H) = \alpha_f(G)\alpha_f(G)\,. \qquad \text{(independence number)}$$

Proof sketch

- Take anti-heptagon. A Pauli realization is:

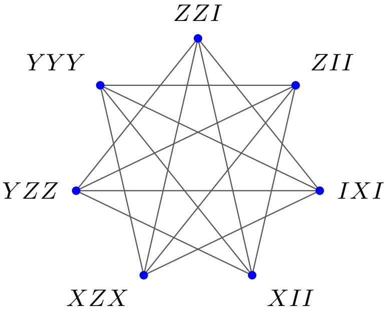


Figure: Anticommutation graph of $\overline{C}_7$ and its Pauli realization.

It has fractional coloring number $\chi_{\mathrm{f}}(\overline{C}_7) = \frac{7}{2}$

(What is its coloring number?)

Proof sketch II

- Take ground state of Hamiltonian

$$H = \sum_{A \in \mathcal{A}} A$$

  Let ϱ be its ground state.
- One can check that on all observables $A \in \mathcal{A}$ the expectation values are equal,

$$|\operatorname{tr}(A\varrho_{\min})| = \frac{(1+2\sqrt{2})}{7} =: a\,.$$

- Define lex product observables and states,

$$\mathcal{A}^m = \mathcal{A}^{\mathrm{lex}\,m} = (\mathcal{A}\,\mathrm{lex}\ldots\mathrm{lex}\,\mathcal{A}) \quad (m \text{ times})\,.$$
$$\varrho_m = \varrho_{\min}^{\otimes K}\,,$$

- Check:

$$|\operatorname{tr}(\varrho_m A)| = a^m \quad \text{for all} \quad A \in \mathcal{A}^m\,.$$

Proof sketch III

- For given m set $\epsilon_m = a^m$.
- Let
$$B_{\epsilon_m}(\varrho_m) = \left\{ P \in \mathcal{P}_n \,:\, |\operatorname{tr}(\varrho_m P)| \geq \epsilon_m \right\},$$
- Realize: $\mathcal{A}^{\mathrm{lex}\, m}$ is a subset of B_{ϵ_m}. (All elements in $\mathcal{A}^{\mathrm{lex}\, m}$ have expectation value ϵ^m or larger)
- Anticommutation graph $G(\mathcal{A}^m)$ is an induced subgraph of $G(B_{\epsilon_m})$, and thus $\chi_{\mathrm{f}}(\mathcal{A}^m) \leq \chi_{\mathrm{f}}(B_{\epsilon_m})$.
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$$\begin{aligned} \chi_{\mathrm{f}}(B_{\epsilon_m}) \cdot \epsilon_m^2 \quad &\geq \quad \chi_{\mathrm{f}}(\mathcal{A}^m) \cdot a^{2m} \\ &= \quad \left(\frac{9 + 4\sqrt{2}}{14}\right)^m \quad \approx \quad (1.046918)^m \,. \end{aligned} \tag{1}$$

Conclusion

- Sampling complexity of the shadow tomography protocol by King et al. scales at least with

$$\begin{aligned} \text{samples:} \quad & \Omega\big(\epsilon^{-2.07598}\log(|S|)/\epsilon^2\big)\,, \\ \text{runtime:} \quad & \Omega\big((T+n^3)\epsilon^{-2.07598}(\log|S|)/\epsilon^2\big)\,. \end{aligned}$$

  thus overall at least with $\Omega(1/\epsilon^{4.07598})$.
- Amplifaction of properties to construct a counterexample is really useful. See also :

  Terence Tao, Amplification, arbitrage, and the tensor power trick `https://terrytao.wordpress.com/2007/09/05/amplification-arbitrage-and-the-tensor-power-trick/`