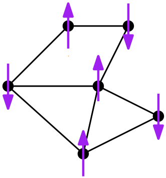


Felix Huber, U. Gdańsk

Second order cone relaxations for quantum Max Cut

arXiv:2411.04120

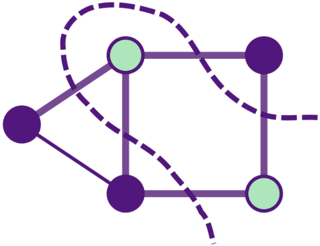
joint work with:
Ojas Parekh & Kevin Thompson, Sandia National Laboratory
Sevag Gharibian, Paderborn University

Max Cut

Let $G = (V, E)$ be a graph. A *cut* is a subset $K \subseteq V$.
Denote by $E(K)$ the edges crossing from K to $\bar{K} = V \backslash K$.

Task: Find a *maximum cut*, so that the the number of edges between the subsets is maximal.

$$\mathrm{MaxCut}(G) \quad = \max_{K \subseteq V} |E(K)|$$



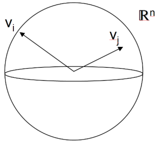
Max Cut II

Some Facts:

- MaxCut is NP-hard.
- Polynomial time approximation algorithm by Goemans and Williamson. Finds cut with approximation ratio

$$\frac{\mathrm{MaxCut}(G)^{\mathrm{approx}}}{\mathrm{MaxCut}(G)^{\mathrm{true}}} \geq 0.878$$

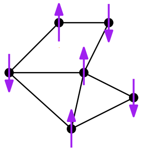
  through randomized rounding of a semidefinite program.
  J. ACM, 42 (6), 1115–1145 (1995)
- GW is the optimal polynomial-time algorithm if the Unique Games Conjecture holds.



Physics of Max Cut

Max Cut is also a physics problem: find ground state energy of system with anti-ferromagnetic interactions (Ising model).

$$\mathrm{MaxCut}(G) \quad = \quad \max \frac{1}{2} \sum_{(ij)\in E} (1 - z_i z_j)$$



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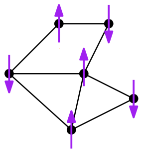


"Quantum Ising model" = quantum Heisenberg model

$$H = -\frac{1}{2} \sum_{(i,j)\in E} (\mathbb{1} - X_i X_j - Y_i Y_j - Z_i Z_j)$$

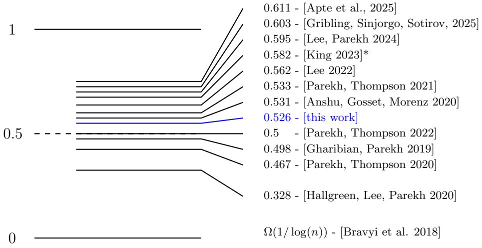
$$\mathrm{QMaxCut(G)} \quad = \quad -\min_{\varrho} \mathrm{tr}(H\varrho)$$

Quantum Max Cut

Quantum Max Cut: find ground state on graph

$$H = -\frac{1}{2} \sum_{(i,j)\in E} (\mathbb{1} - X_i X_j - Y_i Y_j - Z_i Z_j)$$

- QMA-complete.
- Practically challenging: diagonalize matrices of size $2^n \times 2^n$.
- Variational upper bounds / SDP lower bounds.
- Approximation algorithms (NPA + randomized rounding): provides lower bound and upper bound with guarantee.



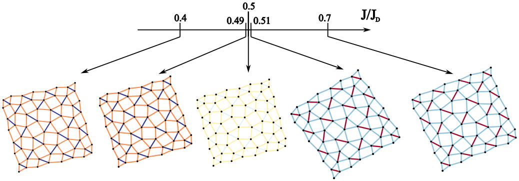
Main result

Previous* approximation algorithms require a semidefinite program of size $n^2 \times n^2$.

SDP of $n \times n$ matrix with second order cone constraints. Approximation ratio of 0.526.

FH, Kevin Thompson, Ojas Parekh, Sevag Gharibian, arXiv:2411.04120

- Factor $\times 10$ in size to previous methods.
- Recovers phases of Shastry-Sutherland model.
- Good scaling (256-qubit systems).



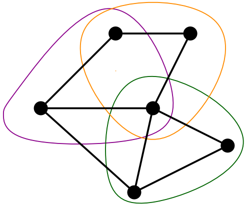
* all entangled approximations.

Key idea

How to obtain a practically useful relaxation + approximation algorithm?

Combine

- Optimization over moment matrix (SDP): Pauli-NPA.
- Optimization over three-qubit marginals (SOCP).



3-qubit marginals

Some facts:

- QMC is real: $H = -\frac{1}{2}\sum_{(i,j)\in E}(\mathbb{1} - X_iX_j - Y_iY_j - Z_iZ_j) = H^T$
- QMC is $SU(2)$ invariant:

$$U^{\otimes n} H (U^{\otimes n})^\dagger = H \qquad \text{for all} \qquad U \in SU(2)$$

How to use this?

- Use the representation theory of the symmetric group / $SU(2)$.
- Define permutation operators $\pi \in S_n$ acting on $(\mathbb{C}^d)^{\otimes n}$:

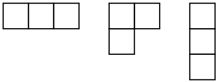
$$\pi \left|v_1\right\rangle \otimes \ldots \otimes \left|v_n\right\rangle = \left|v_{\pi^{-1}(1)}\right\rangle \otimes \ldots \otimes \left|v_{\pi^{-1}(n)}\right\rangle$$

Some Representation theory

- A $U^{\otimes n}$-invariant operator A can be expanded as

$$A = \sum_{\pi \in S_n} a_\pi \pi$$

- The subspaces that are invariant under $SU(2)$ are labeled by *partitions* $\lambda \vdash n$.
- For $n = 3$, there are the partitions $(3), (2,1)$, and $(1,1,1)$. Visually,



- That is, one can block-diagonalize an invariant A.

Three-qubit marginals

Consider a three-qubit marginal of the ground state of H.

$$\varrho_{123} = \sum_{\pi \in S_3} r_\pi \pi \qquad \longleftarrow \qquad \text{six parameters}$$

Further simplifications:

- On $(\mathbb{C}^2)^{\otimes 3}$, the partition $(1,1,1)$ vanishes, i.e.

$$\mathbb{1} - (12) - (13) - (23) + (123) + (132) = 0$$

- If $H = H^T$, then also $\varrho_{ijk}^T = \varrho_{ijk}$.
- Write $\langle \pi \rangle = \operatorname{tr}(\pi \varrho)$.
  Because $(123)^T = (123)^{-1} = (132)$, we have

$$\langle (123) \rangle = \langle (132) \rangle$$

Let's count: We are left with $6 - 1 - 1 - 1 = 3$ parameters.

Three-qubit marginals

$\Longrightarrow$ Describe real $U^{\otimes n}$-invariant three-qubit state by:

$$x = \langle \mathrm{SWAP}_{12} \rangle\,, \qquad y = \langle \mathrm{SWAP}_{23} \rangle\,, \qquad z = \langle \mathrm{SWAP}_{13} \rangle\,,$$

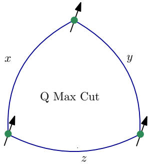
Lieb-Mattis (1962) and Parekh-Thompson (2022) conditions:

$$0 \leq x + y + z \leq 3\,, \tag{LM}$$

$$(x^2 + y^2 + z^2) - 2(xy + xz + yz) + 2(x + y + z) \leq 3\,. \tag{PT}$$

A real $U^{\otimes 3}$-invariant operator on $(\mathbb{C}^2)^{\otimes 3}$ is a state, if and only if LM and PT hold.

FH, Kevin Thompson, Ojas Parekh, Sevag Gharibian, arXiv:2411.04120



Deriving Lieb-Mattis and Parekh-Thompson

A basis for the invariant subspace:

$$
\begin{aligned}
R_+ &= \frac{1}{6}\big(\,\mathrm{id} + (12) + (23) + (13) + (123) + (132)\big)\,,\\
R_- &= \frac{1}{6}\big(\,\mathrm{id} - (12) - (23) - (13) + (123) + (132)\big)\,,\\
R_0 &= \frac{1}{3}\big(2\,\mathrm{id} - (123) - (132)\big)\,,\\
R_1 &= \frac{1}{3}\big(2(23) - (13) - (12)\big)\,,\\
R_2 &= \frac{1}{\sqrt{3}}\big((12) - (13)\big)\,,\\
R_3 &= \frac{i}{\sqrt{3}}\big((123) - (132)\big)\,.
\end{aligned}
$$

Eggeling-Werner, PRA 63, 042111 (2001)

Note:

- R_+ is (up to multiplicities) one-dimensional.
- R_0, R_1, R_2, R_3 behave like Pauli I, X, Y, Z. $\Longrightarrow$ Bloch ball!
- R_- is the anti-symmetric subspace, vanishes on qubits.

Second order cone programming

Second-order cone programs (SOCP) have the form

$$\begin{aligned} \min_x \quad & f^T x \\ \text{subject to} \quad & ||A_i x + b_i|| \leq c_i^T x + d_i \,, \quad i = 1, \ldots, N \,, \end{aligned}$$

Here:

- $||\cdot||$ the Euclidean norm.
- variable: $x \in \mathbb{R}^n$
- problem parameters:
  $f \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{n_i - 1 \times n}$, $b_i \in \mathbb{R}^{n_i - 1}$, $c_i \in \mathbb{R}^n$, and $d_i \in \mathbb{R}$.

. . . In practise, SOCP's are much more efficient to solve than SDP's.

Second order cone relaxation for Quantum Max Cut

Our real invariant three-qubit state constraint is a second order cone constraint!

Combining level-1 from Pauli-NPA and second order constraints together:

$$
\begin{aligned}
\max_{x_{ij}} \quad & \frac{1}{2} \sum_{(i,j)\in E} \left(1 - x_{ij}\right) \\
\text{subject to} \quad & 0 \leq x_{ij} + x_{jk} + x_{ik} \leq 3\,, \\
& (x_{ij}^2 + x_{ik}^2 + x_{jk}^2) - 2(x_{ij}x_{ik} + x_{ij}x_{jk} + x_{ik}x_{jk}) + 2(x_{ij} + x_{ik} + x_{jk}) \leq 3 \\
& M \succeq 0\,, \quad M_{ii} = 3\,, \quad M_{ij} = 2x_{ij} - 1\,.
\end{aligned}
$$

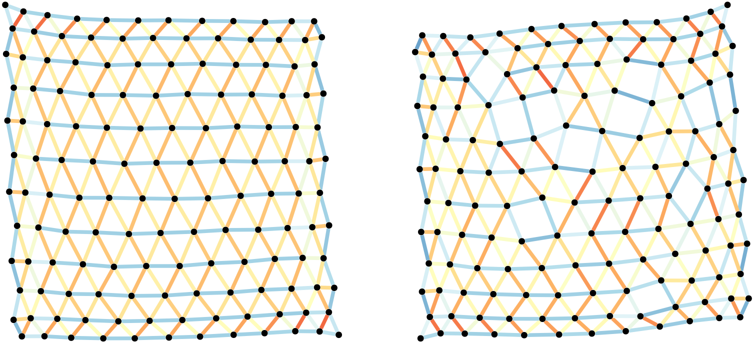
- Upper *and* lower bounds. Efficient in practise.
- Flexible: gives bounds where other methods fail (random graphs, deformed lattices, spin glass).
- However: weak on sparse graphs and highly structured lattices.

Approximation ratio

- Key idea: Take x_{ij} values from level 1 of the Pauli-hierarchy, and compute a maximal matching of G*.
- Set of non-overlapping edges on graph that maximizes the sum of $\sum_{(ij) \in E} x_{ij}$.
- Place singlet for every edge in matching, place product state for unmatched vertices.
- Reasoning with the constraints from LM and PT yields an approximation ratio of 0.526 to the true ground state energy.

*A matching of G is a subset of edges such that no two edges in the subset share a common vertex.

Lower bounds for lattice with defects



- Lower bounds on GS energy.
- cold blue = singlet, red hot = orthogonal to singlet.
- Lattice defects break boundary effects.

Contributions + Outlook

Summary

- New approach for lower bounds: combination of moment matrices and marginal approximations.
- Symmetry reduction is powerful. SOCP are efficient.

Outlook

- Similar analysis for classical-quantum Max Cut
  $H = \sum_{(ij)\in E} J(XX+YY)_{ij} + ZZ_{ij}$, $J \in \mathbb{R}$
- SOCP for arbitrary Hamiltonians with Lovasz bounds.
- Entanglement characterization of classical-quantum 3-RDM

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